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Bounded depth broadcasting

David B. Peters, Joseph G. Peters*

School of Computing Science, Simon Fraser University, Burnaby, B.C., Canada V5A 1S6

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Abstract

Broadcasting is an information dissemination problem in which messages originating at one site of an information network (modelled as a graph) must be transmitted to all other sites as quickly as possible. In this paper we study broadcasting in networks in which information degenerates with each transmission, so there is a limit on the number of times information can be retransmitted before it becomes unusable. We prove lower and upper bounds on the time to broadcast in this setting and on the minimum number of communication links necessary to permit minimum time broadcasting from any originator. We present several general constructions that produce infinite families of optimal networks (minimum time and minimum number of communication links). We also exhibit a number of small optimal networks that are not produced by the general constructions.

Keywords: Broadcasting; Graphs; Communication

1. Introduction

Broadcasting is the process of distributing information from an originator to all other nodes of an information network. In this paper we consider the effects of the degeneration of information that can occur with retransmission in some systems. One common example in which degeneration can occur during the distribution of information is the practice of making photocopies of a report and then distributing the copies while keeping the original. After several generations of photocopying, the copies become unreadable. Degeneration occurs even faster when Fax machines are used to retransmit documents. Perhaps the most common example of imperfect retransmission of information is verbal communication; sometimes a story is almost unrecognizable after it has been retold a few times.

*Corresponding author. E-mail: peters@cs.sfu.ca. Research supported by the Natural Sciences and Engineering Research Council of Canada.

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The problem addressed in this paper is that of constructing broadcast networks, which we model as graphs, under the assumptions that only one piece of information is to be distributed, each communication involves exactly two adjacent nodes and takes one unit of time, no node is involved in two or more simultaneous communications, and there is a constant upper bound on the number of times the information can be retransmitted. In the absence of a bound on retransmissions, at least $\lceil \log_2 n \rceil$ time units are required to complete a broadcast under these assumptions since the number of informed nodes can at most double during each step. When there is a bound on retransmissions, the minimum broadcast time can increase.

In a connected graph G with n nodes, a broadcast originated by a node u determines a spanning tree rooted at u called a *broadcast tree* or *broadcast scheme* for u. The minimum time needed to complete a broadcast originated by node u is the *broadcast time* of u, and the *broadcast time* of G is the maximum of the broadcast times of the nodes of G. If there is no bound on the number of retransmissions, then the minimum broadcast time of any graph on n nodes is $\tau(n) = \lceil \log_2 n \rceil$ and a graph with broadcast time $\tau(n)$ is a *broadcast graph*. The broadcast function $B(n)$ is the minimum number of edges in any broadcast graph with n nodes, and a broadcast graph with n nodes and $B(n)$ edges is called a *minimum broadcast graph* or *mbg*.

Imposing an upper bound d on the number of retransmissions permitted is the same as restricting the depth of broadcast trees to be no more than d. If $d < \lceil \log_2 n \rceil$, then it is no longer possible to double the number of informed nodes during each step, and the minimum broadcast time will increase. Let $v_d(t)$ denote the maximum number of nodes that can be informed in t steps when the depth of broadcast trees is bounded by d. Similarly, we use $\tau_d(n)$ and $B_d(n)$ to denote the depth bounded versions of $\tau(n)$ and $B(n)$.

Lemma 1.1.

$$v_d(t) = \begin{cases} t+1 & \text{if } d = 1, \\ \sum_{i=0}^{d} \binom{t}{i} & \text{if } 1 < d < t, \\ 2^t & \text{if } d \geqslant t. \end{cases}$$

Proof. When $d = 1$, the originator is the only node that can transmit messages. When $d \geqslant t$, the depth bound d imposes no restriction and the number of informed nodes can double during each step. When $1 < d < t$, $v_d(t) = 1 + \sum_{i=0}^{t-1} v_{d-1}(i)$ because a node that is informed in time step i by the originator can inform $v_{d-1}(t-i) - 1$ more nodes in the remaining $t - i$ time steps. This gives the recurrence relation $v_d(t+1) = v_d(t) + v_{d-1}(t)$ which has solution $v_d(t) = \sum_{i=0}^{d} \binom{t}{i}$. $\square$

Corollary 1.1.

$$\tau_1(n) = n - 1,$$

$$\tau_2(n) = \lceil (\sqrt{8n-7} - 1)/2 \rceil,$$

$$\tau_d(n) = \mathrm{O}(n^{1/d}) \quad \textit{if } 1 < d < \lceil \log_2 n \rceil,$$

$$\tau_d(n) = \lceil \log_2 n \rceil \quad \textit{if } d \geqslant \lceil \log_2 n \rceil.$$

Minimum broadcast graphs are difficult to construct and there is no known method for constructing an mbg for arbitrary n. Furthermore, there is no known method for determining $B(n)$ for arbitrary n. In fact, even determining the broadcast time for an arbitrary node in an arbitrary graph is NP-complete (see [12]). There are, however, two known infinite families of mbg's: the hypercubes when $n = 2^k$ [6], and a family of degree $k-1$ Cayley graphs when $n = 2^k - 2$ [9, 4], and mbg's have been found for some small values of n. Since mbg's are so difficult to find, various methods for constructing "sparse" broadcast graphs have been proposed. See [8] for a survey of early work on broadcast graphs and [1] for a recent survey. Generalizations and restrictions of the basic broadcast model that have been investigated include broadcast digraphs [11], a DMA-bound model in which a node can inform a bounded number of neighbours in a single step [10], and bounded degree broadcasting [3]. The bounded degree model is orthogonal to the bounded depth model in the sense that bounded degree increases the depth of broadcast trees whereas bounded depth increases the degree of broadcast trees.

Unfortunately, with the exception of hypercubes, which are mbg's of depth d on $n = 2^d$ nodes, almost none of the results for other models mentioned above are helpful for the bounded depth model. Some of the construction methods for sparse broadcast graphs can be adapted to the bounded depth model, but most of them do not seem to produce very good graphs. Our main goal in this paper is to develop methods for constructing good bounded depth broadcast graphs. We do this in Section 4. In the next section, we exhibit a number of small optimal graphs, most of which are not produced by the general constructions. In Section 3, we prove lower bounds on $B_d(n)$.

2. Bounded depth broadcast graphs

$B_1(n) = \binom{n}{2}$ since a complete graph is required when the broadcast tree is a star. The cases $d = 2$ and $d = 3$ are more interesting. We use the term *(t, d)-broadcast graph* to mean a graph in which a broadcast can be completed from any originator in time t and depth d. The terms *(t, d)-broadcast scheme* and *(t, d)-mbg* are used similarly. Fig. 1 shows several mbg's with bounded depth 2 or 3. To prove that the 6-node graph labelled (3, 2)-mbg is a depth 2 mbg, we first note that at least $\lceil \log_2 6 \rceil = 3$ steps are required. To depth 2 broadcast to five other nodes in three steps, either the originator has degree 3, or the originator has degree 2 and the first node it calls has degree 3. Since every node of degree 2 must be adjacent to a node of degree 3, two nodes of degree 3 are necessary, so $B_2(6) \geqslant 7$. It is an easy exercise to produce depth 2 broadcast trees for each of the three node types to complete the proof. The proofs for most of the other graphs shown in Fig. 1 are also straightforward and are omitted. The

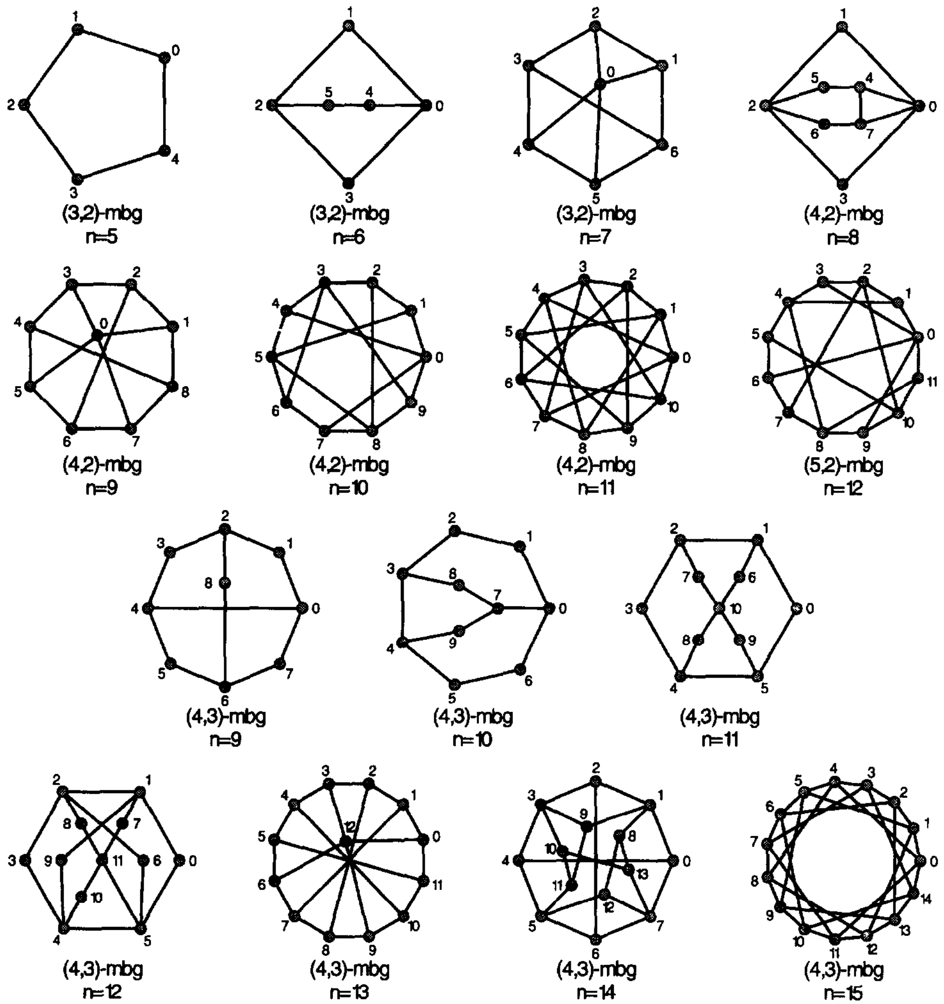

Fig. 1. Minimum broadcast graphs of depth 2 and 3.

exceptions are the 12-node graphs labelled (5, 2)-mbg and (4, 3)-mbg for which explicit proofs follow. We will use the notation δk to refer to a node of degree k in the next two proofs.

Theorem 2.1. $B_2(12) = 21$.

Proof. It takes at least 5 steps to broadcast to 12 nodes with a depth 2 broadcast tree, and the originator must have degree at least 3. A degree 3 originator must be adjacent to either a $\delta 5$ or two $\delta 4$'s.

If a 12 node depth 2 broadcast graph has three or more $\delta 5$'s, then it has at least 21 edges. Every node in a depth 2 broadcast graph is at distance at most 2 from every other node, so if there are only two $\delta 5$'s, then they can satisfy the adjacency requirements of at most nine $\delta 3$'s. This means that there must be at least one $\delta 4$, so there are at least 21 edges.

If the graph has only a single $\delta 5$, then a minimum of two $\delta 4$'s are required in the degree sequence. Since each $\delta 4$ is either adjacent to the $\delta 5$ or at distance 2 from the $\delta 5$, each $\delta 4$ has at most three free edges that are not connected to the $\delta 5$ or its neighbours. Since any $\delta 3$ that is not adjacent to the $\delta 5$ must be adjacent to both $\delta 4$'s, the pair of $\delta 4$'s can satisfy at most three $\delta 3$'s. Since the $\delta 5$ satisfies at most five more $\delta 3$'s, at least one more $\delta 4$ is required and this configuration has 21 edges.

If the graph has only $\delta 3$'s and $\delta 4$'s, then there are at least four $\delta 4$'s since two $\delta 4$'s can satisfy at most seven $\delta 3$'s, and three $\delta 4$'s leave an odd number of odd degree nodes. If there are four $\delta 4$'s, then they have at most 16 edges among them, and each of the eight $\delta 3$'s requires two of the 16 edges. Therefore, no two $\delta 4$'s are adjacent, no $\delta 3$ is adjacent to three $\delta 4$'s, and each pair of $\delta 4$'s has at least one common $\delta 3$ neighbour.

Since each $\delta 3$ is at distance 1 or 2 from each $\delta 4$ (because the depth is 2), we can see that the $\delta 3$'s must occur in pairs such that each $\delta 4$ is adjacent to one of the nodes of each pair. Fig. 2 illustrates one possible configuration that satisfies all of the constraints mentioned so far. Since there are eight $\delta 3$'s and only six pairs of $\delta 4$'s, some pair of $\delta 4$'s (such as nodes 1 and 3 in Fig. 2) must have two common $\delta 3$ neighbours which are not a pair and which are not at distance 1 from each other (nodes 2 and 9 in the figure). One of these (say, 2) cannot be at distance 2 from the mate of the other (11 is the mate of 9), since they have no neighbours in common. This contradicts the depth bound of 2, so at least five $\delta 4$'s are necessary, and 21 edges are required. This establishes that $B_2(12) \geqslant 21$. The 21 edges mbg in Fig. 1 shows that $B_2(12) = 21$. (The broadcast schemes are omitted.) □

Theorem 2.2. $B_3(12) = 17$.

Proof. It takes four steps to broadcast in a depth 3 broadcast tree on 12 nodes, and the originator has degree at least 2. The 12-node graph labelled (4, 3)-mbg in Fig. 1 shows that $B_3(12) \leqslant 17$. Suppose there is a broadcast graph on 12 nodes with only 16 edges. Fig. 3 shows the only possible broadcast tree with a $\delta 2$ originator for such

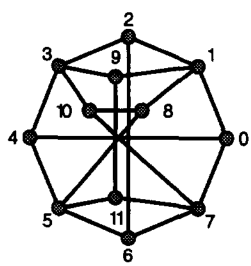

Fig. 2. A 20 edge graph on 12 vertices.

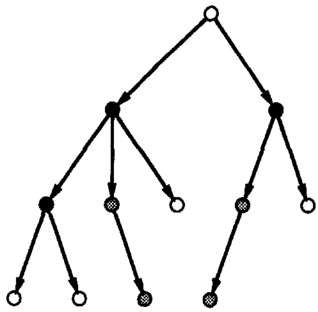

Fig. 3. Depth 3 broadcast tree on 12 nodes with a degree 2 originator.

a graph. From a careful examination of the tree in Fig. 3, it can be determined that at least five nodes of degree 3 or greater are required (the black nodes in Fig. 3 and at least one of each connected pair of grey nodes), that each $\delta 2$ must be adjacent to a node of degree at least 3 and another node of degree at least 4, and that no $\delta 2$ may be part of a cycle of length less than 5 unless it is adjacent to two nodes with degree at least 4. It follows that there are at least six $\delta 2$'s.

If there were a node of degree at least 3 that was only adjacent to $\delta 2$'s, it could not broadcast with depth 3 to more than 11 nodes in the four available time steps. As a result, every node must have a neighbour of degree at least 3. Since there are at least 5 nodes with degree 3 or greater, at least 3 edges are needed to ensure that each of these nodes has a neighbour with degree at least 3. This leaves at most 13 edges that may be incident on a $\delta 2$. Since no two $\delta 2$'s share an edge, the broadcast graph has at most six $\delta 2$'s. The only possible degree sequence is two $\delta 4$'s, four $\delta 3$'s, and six $\delta 2$'s.

Since each $\delta 2$ must be adjacent to at least one $\delta 4$, and at most three $\delta 2$'s may be adjacent to any one $\delta 4$, the only possibility is that each $\delta 2$ is adjacent to exactly one $\delta 4$ and each $\delta 4$ is adjacent to exactly three $\delta 2$'s. Each of the six $\delta 2$'s must be also adjacent to a $\delta 3$. Since there are only four $\delta 3$'s, there is a $\delta 3$, which we will call node 0, that is adjacent to two $\delta 2$'s. Furthermore, since no $\delta 2$ in this configuration may be a part of a 4-cycle, the $\delta 2$'s adjacent to node 0 must be adjacent to different $\delta 4$'s. The third neighbour of node 0 cannot be a $\delta 4$, for that would cause a $\delta 2$ to be part of a triangle.

If we broadcast from node 0, we get the broadcast tree shown on the left in Fig. 4. Node 1 is a $\delta 3$, and nodes 6 and 7 are the $\delta 4$'s since they are the only other nodes adjacent to the $\delta 2$'s labelled 2 and 3. Node 5 cannot be a $\delta 2$ since this would force node 10 to be a third $\delta 4$. So the broadcast graph must contain a $\delta 3$ (node 1) that is adjacent to the other three $\delta 3$'s (nodes 0, 4, and 5). As a result, the remaining three $\delta 3$'s must each be adjacent to two $\delta 2$'s. This uses 15 edges, and leaves both $\delta 4$'s missing an edge; the last edge must connect the $\delta 4$'s giving the graph on the right in Fig. 4. The topmost node of the graph does not have a depth 3 broadcast scheme that uses four steps contradicting the assumption that there is a broadcast graph with 16 edges. Therefore, $B_3(12) \geqslant 17$. $\square$

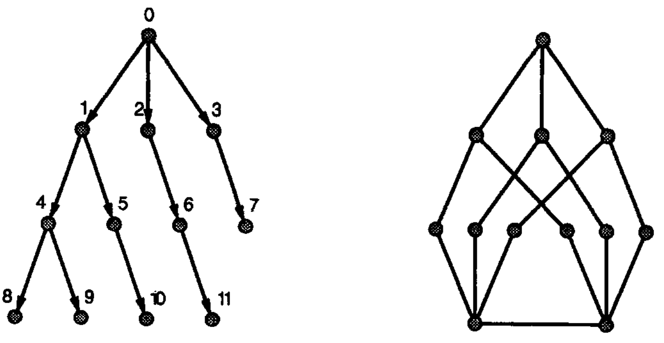

Fig. 4. Broadcast tree for an originator of degree 3 and resulting graph.

3. Lower bounds

Lemma 3.1. *If $n = v_d(t)$ for some t and d, then $B_d(n) \geqslant \lceil \frac{1}{2}tn \rceil$ and every node has degree at least t.*

Proof. A complete broadcast tree of depth d is needed to broadcast to $v_d(t)$ nodes, so the originator must have degree at least t. □

Lemma 3.2. *If $n = v_d(t) - 2$ for some t and some $d \geqslant 2$, then $B_d(n) \geqslant \lceil \frac{1}{2}(t-1)n \rceil$ and every node has degree at least $t - 1$.*

Proof. The broadcast tree for $v_d(t)$ nodes has exactly two nodes of degree t – the originator, and the first child. Both of these nodes have a neighbour which is a leaf in the broadcast tree, namely the nodes they inform in the last time step. If these two leaves are removed, $v_d(t) - 2$ nodes remain, and no node has degree greater than $t - 1$. If $d \geqslant 2$, then at least 3 nodes must be removed from the original broadcast tree to reduce the degree of the originator to $t - 2$. Thus, an originator of degree $t - 1$ is necessary and sufficient to broadcast to $v_d(t) - 2$ nodes. □

Lemma 3.3. *If $n = v_d(t) - 1$ for some t and d, then $B_d(n) \geqslant \lceil \frac{1}{2}(nt^2/(t+1)) \rceil$.*

Proof. For any arbitrary broadcast graph with n nodes, let n_i denote the number of nodes of degree i and let $\delta_{\max}$ be the maximum degree of any node. To broadcast to $v_d(t) - 1$ nodes, an originator of degree at least $t - 1$ is necessary, and either the originator or the first node it informs must have degree at least t. The number of edges in the graph is therefore $\lceil \frac{1}{2}((t-1)n_{t-1} + \sum_{i=t}^{\delta_{\max}} i \cdot n_i) \rceil$. Furthermore, $\sum_{i=t}^{\delta_{\max}} i \cdot n_i \geqslant n_{t-1}$, and $\sum_{i=t}^{\delta_{\max}} i \cdot n_i \geqslant t(n - n_{t-1})$. The intersection of the latter two constraints gives a lower bound of $nt/(t+1)$ on $\sum_{i=t}^{\delta_{\max}} i \cdot n_i$. The lower bound on $B_d(n)$ follows. □

The approach used in the three lemmas above can be generalized to get bounds for other values of n, d, and t. The basic idea is to solve an integer programming problem in which the constraints are inequalities relating n and several n_i's where n_i is the number of nodes of degree i. Unfortunately, to obtain closed form solutions, we had to assume that there is an mbg with maximum degree $\tau_d(n)$ for every choice of n and d. All graphs that we have investigated support this conjecture and we are confident that it is true, but we have not discovered a proof.

Conjecture 3.1. For any values n and d there is an mbg with maximum degree $\tau_d(n)$.

Conjecture 3.2. If $n = v_d(t) - 4$ for some t and d, then $B_d(n) \geqslant \lceil \frac{1}{2}(nt(t-1)/(t+1)) \rceil$. If $n = v_d(t) - 3$ for some t and d, then $B_d(n) \geqslant \lceil \frac{1}{2}((2t^2 - 3t + 1)n/(2t+1)) \rceil$.

4. Families of broadcast graphs

In this section, we present several general construction methods for producing depth bounded broadcast graphs. The first method is a slightly modified version of a node deletion method due to Wang (see [2] or [13]).

Theorem 4.1. *If there is a (t, d)-broadcast graph with $n \neq 2^j + 1$ nodes, e edges, and a node with degree δ, then there is a (t, d)-broadcast graph on $n - 1$ nodes with at most $e - \delta + \binom{\delta}{2}$ edges.*

Proof. Let G be a graph satisfying the statement of the lemma and let v be a node of degree δ. Form the graph $G\backslash v$ by deleting v and its incident edges from G and then adding all edges necessary to form a clique among the former neighbours of v. Consider any broadcast scheme for G not originating at v and let v_0 be the node that informs v. Suppose that v_0 informs v at time t_0 and that v informs its children $v_1, v_2, \ldots, v_j$ at times $t_0 + 1, t_0 + 2, \ldots, t_0 + j$ where $j < \delta$. If v was informed at depth d_0, then all of its children are informed at depth $d_0 + 1$. We can modify the broadcast scheme for G to work on $G\backslash v$ by adding a call from v_0 to v_j at time t_0 and depth d_0 and a call from v_j to each other v_i, $1 \leqslant i < j$, at time $t_0 + i$ and depth $d_0 + 1$. v_j is informed earlier and at smaller depth than it was in G and is ready to broadcast to its other neighbours one step sooner (at time $t_0 + j$). All other former children of v are informed at the same time and depth as they were in G. $\square$

The following theorem uses a construction similar to the *two-way split* in [5]. If $G = (V, E)$ and $G' = (V', E')$ are two graphs on n and n' nodes, respectively, with $n \geqslant n'$, and ϕ is a surjection $\phi : V \xrightarrow{\text{onto}} V'$, then the graph $G +_\phi G'$ is constructed from G and G' by adding the edge $(v, \phi(v))$ for every node $v \in V$.

Theorem 4.2. *Let $G = (V, E)$ be a (t, d)-broadcast graph with n nodes and $G' = (V', E')$ a $(t - 1, d - 1)$-broadcast graph with at least $n/2$ nodes. If ϕ is a surjection $\phi: V \xrightarrow{\text{onto}} V'$ with $|\phi^{-1}(w)| \leqslant 2$ for all nodes $w \in V'$, then $G +_\phi G'$ is a $(t + 1, d)$-broadcast graph.*

Proof. If the originator u is in G, then u first calls its neighbour $\phi(u) \in V'$. u and $\phi(u)$ then complete the broadcasts in G and G'. This broadcast scheme uses time $t + 1$ and depth d. If the originator u is in G', then u first completes a broadcast in G' in time $t - 1$ and depth $d - 1$. Each node in G' has at most two neighbours in G, so all nodes of G can be informed in two more steps and depth d. □

The next two theorems use hypercubes that have been augmented with an edge between each pair of *antipodal nodes*, that is, nodes with labels that are bit-wise complements. We use Q_k to denote a k-dimensional hypercube and Q_k^+ to denote an *augmented hypercube* on 2^k nodes.

Theorem 4.3. *Q_{2d}^+ is a $(2d + 1, d)$-mbg.*

Proof. Without loss of generality, assume that the originator is $\vec{0}$ (i.e., all bits of the label of the originator are 0's). In the first step, $\vec{0}$ calls its antipodal neighbour $\vec{1}$. In the remaining $2d$ steps, both $\vec{0}$ and $\vec{1}$ use the usual dimension by dimension hypercube broadcast scheme. If we truncate the resulting broadcast tree at level d, then the first level contains all nodes with binary representations containing either one 1, or $2d$ 1's. The second level contains all nodes with either two or $2d - 1$ 1's, and so on. At level d, all nodes with either d or $d + 1$ 1's occur. Since this tree includes every node of Q_{2d}^+ at some level, it is a depth d, time $2d + 1$ broadcast tree for Q_{2d}^+. By Lemma 1.1, $v_d(2d + 1) = \sum_{i=0}^{d} \binom{2d+1}{i} = \frac{1}{2}(2^{2d+1}) = 2^{2d}$, *so a* $(2d + 1, d)$-broadcast graph has minimum degree $2d + 1$ by Lemma 3.1. Therefore, Q_{2d}^+ is an mbg. □

Theorem 4.4. *Q_{2d-1}^+ is a $(2d, d)$-broadcast graph.*

Proof. If we use the same construction as in Theorem 4.3, then nodes with d 1's in their binary representations occur at level d as descendants of both $\vec{0}$ and $\vec{1}$. If the duplicate nodes are deleted from the subtree rooted at $\vec{1}$, then the resulting tree is a $(2d, d)$-broadcast scheme for Q_{2d-1}^+. □

In the remainder of this section, we present constructions based on Cayley graphs. We use $X(n, S)$ to denote a Cayley graph with nodes $\{0, 1, \ldots, n - 1\}$ and generators S. For example, in $X(n, \{\pm 1, \pm 4\})$, each node i is adjacent to nodes $i + 1, i - 1, i + 4$, and $i - 4$, where all arithmetic on indices is mod n. We will also use a class of augmented Cayley graphs with an odd number of generators which we denote $X^+(n, S)$. In $X^+(n, S)$, all generators except the last occur in pairs. The last "generator" only adds the minimum number of edges necessary to ensure that each node has an incident edge of this type. For example, in $X^+(31, \{\pm 1, \pm 4, 16\})$, there are 16 edges of the form $(i, i + 16)$. Each node i is adjacent to nodes $i + 1, i - 1, i + 4, i - 4$, and either $i + 16$ or $i - 16$. One node j is adjacent to both $j + 16$ and $j - 16$; it does not matter which node.

Theorem 4.5. *$X(2^{d+1} - 1, \{\pm 2^0, \pm 2^2, \pm 2^4, \ldots, \pm 2^{d-1}\})$ is a $(d + 1, d)$-mbg for d odd.*

Proof. By Lemma 1.1, $v_d(d + 1) = \sum_{i=0}^{d} \binom{d+1}{i} = 2^{d+1} - 1$, so we know from Lemma 3.1 that each originator of a broadcast must have degree at least $d + 1$. $X(2^{d+1} - 1, \{\pm 2^0, \pm 2^2, \pm 2^4, \ldots, \pm 2^{d-1}\})$ meets this bound, so to prove that it is a $(d + 1, d)$-mbg, we must show that it is a $(d + 1, d)$-broadcast graph. The theorem is clearly true when $d = 1$. We prove the result for odd $d > 1$ by establishing the following two properties about broadcasts from node 0.

- Node 0 of $X(n, \{\pm 2^0, \pm 2^2, \pm 2^4, \ldots, \pm 2^{2k-2}\})$ can inform all nodes in the range $[-(2^{2k} - 1)/3, (2^{2k+1} - 2)/3]$ in time $2k$ and depth $2k$.
- Node 0 of $X(n, \{\pm 2^0, \pm 2^2, \pm 2^4, \ldots, \pm 2^{2k-2}\})$ can inform all nodes in the range $[-(2^{2k} - 1)/3, (2^{2k+1} - 2)/3 - 1]$ in time $2k$ and depth $2k - 1$.

Note that neither of the ranges wraps around when $n > 2^{2k} - 1$ and $k \geqslant 1$. That is, $n - a > b$ for each range $[-a, b]$. The two properties will be proved together by induction.

The basis has $k = 1$. It is clear that node 0 can inform all nodes in the range $[-1, 2]$ in time 2 and depth 2 in any cycle of size $n > 3$. If the depth is limited to 1, then nodes in the range $[-1, 1]$ can be reached.

Now, assume that for $k \geqslant 1$ and $n > 2^{2k} - 1$, node 0 of $X(n, \{\pm 2^0, \pm 2^2, \pm 2^4, \ldots, \pm 2^{2k-2}\})$ can inform all nodes in the range $[-(2^{2k} - 1)/3, (2^{2k+1} - 2)/3]$ in time $2k$ and depth $2k$, and all nodes in the range $[-(2^{2k} - 1)/3, (2^{2k+1} - 2)/3 - 1]$ in time $2k$ and depth $2k - 1$.

Consider the Cayley graph $X(n, \{\pm 2^0, \pm 2^2, \pm 2^4, \ldots, \pm 2^{2k-2}, \pm 2^{2k}\})$ and allow time $2k + 2$. Node 0 can use its two "longest" edges in the first two steps to call 2^{2k} and then -2^{2k} and node 2^{2k} can use one of its longest edges to call 2^{2k+1} in step 2. We now use the induction hypothesis for each of the four nodes that is informed at the end of the second step. We consider the cases of depth $2k + 2$ and depth $2k + 1$ together.

Node -2^{2k} was informed in step 2 and depth 1. By induction, in the remaining $2k$ steps it can inform all nodes in the range $[-(2^{2k} - 1)/3 - 2^{2k}, (2^{2k+1} - 2)/3 - 2^{2k}]$ in additional depth $2k$. Node 0 has $2k$ steps remaining, so it can inform all nodes in $[-(2^{2k} - 1)/3, (2^{2k+1} - 2)/3]$ in total depth $2k$. Node 2^{2k} was informed in step 1 at depth 1 and used step 2 to call node 2^{2k+1}. It therefore has $2k$ steps and depth at least $2k$ remaining, so it can inform all nodes in $[-(2^{2k} - 1)/3 + 2^{2k}, (2^{2k+1} - 2)/3 + 2^{2k}]$. Node 2^{2k+1} was informed in step 2 at depth 2. It therefore has $2k$ steps and either depth $2k$ or $2k - 1$ remaining. By induction, in the remaining $2k$ steps it can inform all nodes in $[-(2^{2k} - 1)/3 + 2^{2k+1}, (2^{2k+1} - 2)/3 + 2^{2k+1}]$ with additional depth $2k$, and all nodes in $[-(2^{2k}-1)/3+2^{2k+1}, (2^{2k+1}-2)/3+2^{2k+1}-1]$ with additional depth $2k-1$. Since these ranges for the four nodes are adjacent and disjoint, node 0 can broadcast to all nodes in $[-(2^{2k+2}-1)/3, (2^{2k+3}-2)/3]$ in time $2k+2$ and depth $2k+2$, and to all nodes in $[-(2^{2k+2}-1)/3, (2^{2k+3}-2)/3-1]$ in time $2k+2$ and depth $2k+1$. This proves the two properties.

To prove that $X(2^{d+1}-1, \{\pm 2^0, \pm 2^2, \pm 2^4, \ldots, \pm 2^{d-1}\})$ is a $(d+1, d)$-broadcast graph, let node 0 call 2^{d-1} in step 1, and -2^{d-1} in step 2, while node 2^{d-1} calls 2^d in step 2. By the two properties, these four nodes can broadcast within the remaining steps and depth available to them to ranges that are disjoint and that collectively include all nodes of the graph. □

Theorem 4.6. *$X^+(2^{d+1}-1, \{\pm 2^0, \pm 2^2, \pm 2^4, \ldots, \pm 2^{d-2}, 2^d\})$ is a $(d+1, d)$-mbg for d even.*

Proof. By the same argument as in the proof of Theorem 4.5, the graph has the right number of edges to be an mbg. Node 0 informs 2^d or -2^d in the first step of a $(d+1, d)$-broadcast scheme. Nodes 0 and 2^d or -2^d can now inform two ranges that include all other nodes within the available time and depth by the same argument as in the proof of Theorem 4.5. Since every node i has a neighbour $i+2^d$ or $i-2^d$, there is a $(d+1, d)$-broadcast scheme for every originator. □

The construction methods used in Theorems 4.5 and 4.6 can be used to construct other mbg's and sparse broadcast graphs for specific values of n, t, and d. We will concentrate on graphs for which $t = d+2$, but the techniques will work for almost any choices of t and d. Both constructions use the notion of *ranges* of nodes of Cayley graphs or argumented Cayley graphs which can be informed by the originator within certain time and depth bounds. We will say that $[-a, b]$ is a *feasible range* for a Cayley graph or augmented Cayley graph with generator set S for specific t and d if node 0 can inform all nodes in the range in t steps and depth d. For all of the graphs that we will consider, when $[-a, b]$ is a feasible range, $[-b, a]$ will also be a feasible range. We will start with a set of "basis" ranges and use the following fact to construct larger ranges.

Observation 4.1. *If $[-a, b]$ is a feasible range for S, t, and d, and $[-e, f]$ is a feasible range for S, t, and $d+1$, then $[-(a+b+e+1), f]$ is a feasible range for $S \cup \{b+e+1\}$, $t+1$, and $d+1$.*

It is easy to show that for $t = 4$ and the set of generators $\{\pm 1, \pm 4\}$, $[-5, 5]$ is a feasible range when $d = 2$, $[-5, 9]$ and $[-6, 8]$ are feasible ranges when $d = 3$, and $[-5, 10]$, $[-6, 9]$, $[-7, 8]$ are feasible ranges when $d = 4$. When applying Observation 4.1 to this set of basis ranges, the degree of the resulting graphs can be minimized by finding generators for which the observation can be applied two consecutive times,

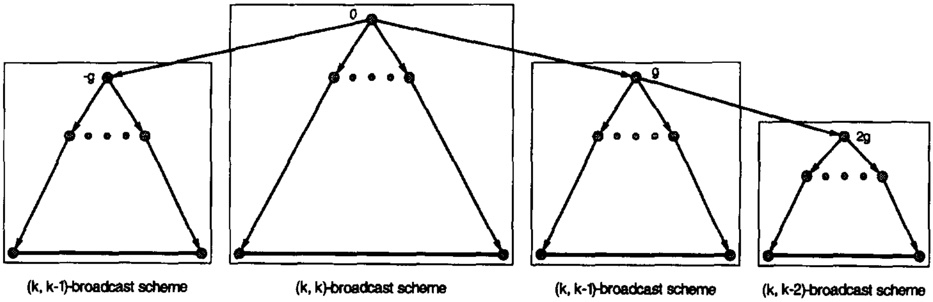

Fig. 5. $(k + 2, k)$-broadcast scheme using two applications of a single generator g.

as illustrated in Fig. 5. In fact, this is exactly what Theorems 4.5 and 4.6 do. The construction in Theorem 4.5 uses the generator 2^{d-1} twice to reach node 2^d from node 0, and the other generators are used in the same way for smaller ranges. The construction of Theorem 4.6 is similar after the first step. Applying Observation 4.1 gives the following results. All of the graphs are mbg's because each is a (t, d)-broadcast graph on $v_d(t)$ nodes and must therefore have minimum degree at least t by Lemma 3.1.

- $X(11, \{\pm 1, \pm 4\})$ is a (4, 2)-mbg.
- $X(26, \{\pm 1, \pm 4, 15\})$ is a (5, 3)-mbg.
- $X(57, \{\pm 1, \pm 4, \pm 15\})$ is a (6, 4)-mbg.
- $X(120, \{\pm 1, \pm 4, \pm 15, 62\})$ is a (7, 5)-mbg.
- $X(247, \{\pm 1, \pm 4, \pm 15, \pm 62\})$ is a (8, 6)-mbg.

When the initial set of basis ranges is augmented with some larger feasible ranges, more mbg's result. For example, the ranges $[-6, 19]$, $[-8, 17]$, and $[-9, 16]$ are feasible for $\{\pm 1, \pm 4, 15\}$, $t = 5$, and $d = 3$. This larger basis gives the following mbg's.

- $X(120, \{\pm 1, \pm 4, \pm 15, 64\})$ is a (7, 5)-mbg.
- $X(247, \{\pm 1, \pm 4, \pm 15, \pm 64\})$ is a (8, 6)-mbg.
- $X(502, \{\pm 1, \pm 4, \pm 15, \pm 64, 255\})$ is a (9, 7)-mbg.
- $X(1013, \{\pm 1, \pm 4, \pm 15, \pm 64, \pm 255\})$ is a (10, 8)-mbg.

5. Summary of results

Table 1 shows the values of $v_d(t)$ for small values of t and d. The mbg's corresponding to the column $d = 1$ are complete graphs with $t + 1$ nodes. The entries along the

Table 1
Values of $v_d(t)$ for small t and d

Time t	Depth $d=1$	2	3	4	5	6	7	8	9
1	2	2	2	2	2	2	2	2	2
2	3	4	4	4	4	4	4	4	4
3	4	7	8	8	8	8	8	8	8
4	5	11	15	16	16	16	16	16	16
5	6	16	26	31	32	32	32	32	32
6	7	22	42	57	63	64	64	64	64
7	8	29	64	99	120	127	128	128	128
8	9	37	93	163	219	247	255	256	256
9	10	46	130	256	382	466	502	511	512

diagonal $t = d$ are powers of 2 and the hypercubes are mbg's corresponding to these entries. The entries along the subdiagonal $t = d + 1$ are of the form $2^t - 1$. The Cayley graphs and augmented Cayley graphs from Theorems 4.5 and 4.6 are mbg's corresponding to these entries. Theorem 4.3 gives one more infinite family of mbg's corresponding to the entries with $t = 2d + 1$. Only four entries, (3, 1), (5, 2), (7, 3), and (9, 4), of this type appear in the table.

Table 2 summarizes most of the lower and upper bounds from Sections 2 and 4 that do not result from the general constructions. All of the lower bounds are degree sequence arguments of the same type as the proofs of Theorems 2.1 and 2.2. (Most of these arguments are omitted.) The values marked with superscript a's are the mbg's shown in Fig. 1, except the values for $n = 4$, $d = 2$, and $n = 8$, $d = 3$, which are hypercubes, and the value for $n = 5$, $d = 3$ which results from Observation 4.1. The values marked by a superscript b used the construction of Theorem 4.2. Some of the graphs that give the other upper bounds in the table are shown in Fig. 6. The graph in Fig. 6 for $n = 14$, $d = 2$ is $X(14, \{\pm 1, \pm 4, 7\})$ and the graph for $n = 22$, $d = 3$ is $X(22, \{\pm 1, \pm 4, 11\})$. Most of the graphs corresponding to the remaining upper bounds in the table are also Cayley graphs. In particular, $X(18, \{\pm 1, \pm 2, \pm 4, 9\})$, $X(19, \{\pm 1, \pm 2, \pm 4, \pm 7\})$, $X(20, \{\pm 1, \pm 2, \pm 4, \pm 7, 10\})$, and $X(22, \{\pm 1, \pm 2, \pm 4, \pm 7, \pm 13\})$ are depth 2 broadcast graphs, and $X(24, \{\pm 1, \pm 4, \pm 7\})$ and $X(26, \{\pm 1, \pm 4, \pm 7, \pm 11\})$ are depth 3 broadcast graphs. (In fact, some of the mbg's in Fig. 1 are also Cayley graphs.)

One more type of construction accounts for the three remaining upper bounds in the table. The graphs for $n = 15$, $d = 2$, and $n = 17$, $d = 2$ in Fig. 6 and some of the graphs in Fig. 1 are also of this type. As an example, the graph for $n = 15$, $d = 2$ is formed from the Cayley graph for $n = 14$, $d = 2$ by choosing three of the longest edges which are as equally spaced as possible around the graph and inserting the 15th node in the middle of them. The graph for $n = 17$, $d = 2$ also splits three long edges. The graph for $n = 21$, $d = 2$ is formed by splitting five long edges of

Table 2

Lower and upper bounds

Depth 2								Depth 3							
n	$\tau_2(n)$	$B_2(n)$		n	$\tau_2(n)$	$B_2(n)$		n	$\tau_3(n)$	$B_3(n)$		n	$\tau_3(n)$	$B_3(n)$	
		Lower	Upper			Lower	Upper			Lower	Upper			Lower	Upper
4	2	4	4[a]	14	5	28	35	8	3	8	8[a]	18	5		33
5	3	5	5[a]	15	5	32	37	9	4	11	11[a]	19	5		40[b]
6	3	7	7[a]	16	5	40	40[a]	10	4	12	12[a]	20	5		44[b]
7	3	11	11[a]	17	6	32	59	11	4	14	14[a]	21	5		48[b]
8	4	11	11[a]	18	6	38	63	12	4	17	17[a]	22	5		55
9	4	14	14[a]	19	6	41	76	13	4	20	20[a]	23	5		58
10	4	17	17[a]	20	6	50	90	14	4	23	23[a]	24	5	48	72
11	4	22	22[a]	21	6	54	95	15	4	30	30[a]	25	5	53	87
12	5	21	21[a]	22	6	66	110	16	5		26	26	5	65	65[a]
13	5	23	30					17	5		32[b]				

[a] mbg's.

[b] Broadcast graphs from Theorem 4.2.

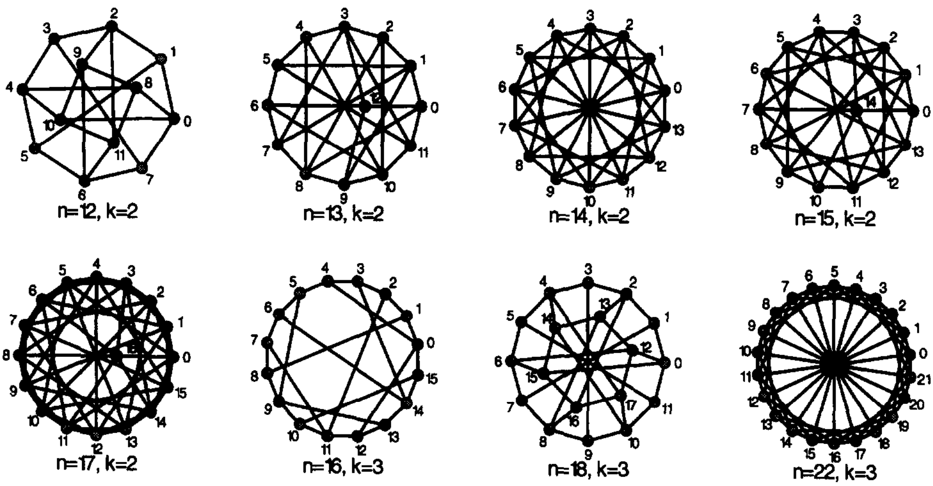

Fig. 6. Some sparse broadcast graphs.

$X(20, \{\pm 1, \pm 2, \pm 4, \pm 7, 10\})$, the graph $n = 23$, $d = 3$ is formed by splitting three long edges of $X(22, \{\pm 1, \pm 4, 11\})$, and the graph for $n = 25$, $d = 3$ is formed by splitting three long edges of $X(24, \{\pm 1, \pm 4, \pm 7, 12\})$. Note that the graphs for $n = 12$, $d = 2$, and $n = 18$, $d = 3$ in Fig. 6 also involve the splitting of long edges of Cayley graphs although in a more complex way.

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